# Basis v 2024.09.24 Fontys ICT

## Axioms for the real numbers

There is a binary operation called “addition”, written “+” (infix) such that for all numbers x,y,z the following hold:

1. x + (y + z) = (x + y) + z (*associativity*)
2. x + y = y + x (*commutativity*)
3. 0 + x = x (addition has a *neutral element*, called 0)
4. For every x there exists a number s such that x + s = s + x = 0. This number is written as –x, leading to:  
   x + –x = –x + x = 0 (every number has an *additive inverse*)

x – y is shorthand for x + (– y)

There is a binary operation called “multiplication”, written “.” (infix) such that for all numbers x,y,z the following hold:

1. x.(y.z) = (x.y).z (*associativity*)
2. x.y = y.x (*commutativity*)
3. 1.x = x (multiplication has a *neutral element*, called 1)
4. If x ≠ 0 then there exists a number s such that x.s = s.x = 1. This number is written as 1/x, leading to:  
   x.(1/x) = (1/x).x = 1 (every number except 0 has a *multiplicative inverse*)
5. x.(y + z) = x.y + x.z (*distributivity, distributive law*)

x.y is sometimes also written as x × y or xy.  
x/y is shorthand for x.(1/y)  
1/x is sometimes also written as x-1

x/y is sometimes also written as , or as x ÷ y.

x2 is shorthand for x.x, x3 for x.x.x, etc.  
x-2 is shorthand for 1/x2, x-3 for 1/x3, etc.

(Order axioms)

There is a binary relation “≤” such that for all numbers x,y,z:

1. if x ≤ y and y ≤ z then x ≤ z (*transitivity*)
2. if x ≤ y and y ≤ x then x = y (*anti-symmetry*)
3. x ≤ x (*reflexivity*)
4. either x ≤ y or y ≤ x (dichotomy, ≤ is a *total* *ordering*)
5. if x ≤ y, then x + z ≤ y + z (*translation-invariance*)
6. if 0 ≤ x and 0 ≤ y, then 0 ≤ x.y   
    (*the non-negative numbers are closed under multiplication)*

x < y is short for “x ≤ y ∧ ¬(x = y)”, it is called the/a *strict* order relation, from which the *ordinary* or *non-strict* order relation ≤ can be reconstructed as “x < y ∨ x = y”.  
x ≥ y is another way to write y ≤ x, x > y is another way to write y < x.   
x < y < z is short for “x < y ∧ y < z”; and similar for >, ≤, ≥, …

|x| is notation for: x if x ≥ 0, –x otherwise.

Non-triviality axiom: ¬(0 = 1)

## Proofs with these axioms

**Some examples**

**Theorem** (“*cancellation law*”): *For all real numbers a, b, c: if a + c = b + c, then a = b.*  
**Proof**: If a + c = b + c, then (a + c) + (–c) = (b + c) + (–c)  
Hence (by associativity), a + 0 = b + 0  
Hence (because 0 is neutral for addition), a = b  
∎ *(🡸 this is an ‘end-of-proof’ symbol)*

Alternative write-up of the same argument:   
If a + c = b + c, then a = a + 0 = a + (c + –c) = (a + c) + –c = (b + c) + –c = b + (c + –c) = b + 0 = 0.  
  
Such chains of equalities are allowed because of the following ***axioms of equality*:**

1. if x = y and y = z, then x = z (*transitivity*)
2. if x = y then y = x (*symmetry*)
3. for every x, x = x (*reflexivity*)

The above theorem is sometimes called right-cancellation for abvious reasons. It will be clear that the analogous left-cancellation theorem (*if c + a = c + b, then a = b*) can proven in a similar way, or by appeal to right-cancellation plus the commutative law for addition.

**Theorem**: *For every real number a: a.0 = 0*   
**Proof**: For every real number a: a.0 + 0 = a.0 = a.(0+0) = a.0 + a.0 (because of distributivity)  
This implies a.0 = 0 because of the cancellation laws proven previously (plus symmetry of equality).  
∎

One reason why 0 can’t have a multiplicative inverse can now be seen as follows: if there were a number s such that s.0 = 1, then for any number x, we would have:

x = 1.x = (s.0).x = s.(0.x) = s.0 = 0  
(This is also the reason for the non-triviality axiom.)

We can now also see that 0 < 1: if that were not so, dichotomy would give 1 ≤ 0, from which translation-invariance would give 0 ≤ –1, hence 0 ≤ (–1).(–1) (because of order axiom iv), hence: 0 ≤ 1, which together with 1 ≤ 0 would give 0 = 1 (because of anti-symmetry of ≤).

## Some notes before starting the exercises

It might be nice to do the intuition visualizes

## Exercises

Prove (for all real numbers a,b,c) :

1. –a = 0 – a

**Intuition**

# TODO: herman gaf mij de tip om eerst een problem analyze te doen, dus bij deze

The problem at hand consists of the following:

* The neutral element for addition (i.e. 0)
* Subtracting some real number a from the neutral element (i.e. 0 – a)
* The negative version of a, also known as the negative inverse
* A statement that says subtracting a from 0 is equal to negative a (-a = 0 – a)

Side note: real numbers are numbers with infinite possible decimals. In other words:

Let the example set be S = { 0, 0.1, 0.11 … 1, 1.1, 1.11, ... }, then S is a subset of ℝ

Or more formally notated as S ⊆ ℝ

Intuitively this makes sense when viewing it from a number line. Subtraction when done on a number line is the same as moving left from some starting point by some number of steps. In this case the starting point is 0 and we move left by a step.

Side note: using the number line here for intuition makes sense as we only move in one dimension.

<insert visualization of a line with numbers above it here>

The above describes what happens after the equality sign in the problem, i.e. 0 – a. Now let’s understand what happens before the equality sign. Not much to be honest :P there is simply the negative inverse for the given real number a.

Actually, now that I think about it, there is more happening then meets the eye. Let’s start with this: when is any given number considered negative? Or in other words: what does it mean for any given number to be negative? To answer this question, we need to look at the axiom of binary relation between numbers: less than or equal to, aka ≤.

In this axiom, it is defined that for any number x and y, where 0 ≤ x and 0 ≤ y, quote: *“0 ≤ x.y (the non-negative numbers are closed under multiplication)*”. This gives us the hint that when 0 ≤ x and 0 ≤ y, both x and y are non-negative when … (see the rest of the proof). Using this we know that when x is less than 0, it is considered negative. The same goes for y. In other words, any number x is negative when 0 > x. (notice we use > to more or less flip the ≤)

TODO: now incorporate all the above into 1 intuitively understandable phrase

**Proof**

# TODO: loop omhoog in plaats van naar beneden

Hence, -a = 0 - a

∎

1. (–1). (–1) = 1
2. If a > 0, then –a < 0
3. a. If a < b and c > 0, then a.c < b.c  
   b. If a < b and c < 0, then a.c > b.c  
   c. If a ≤ b and c > 0, then a.c ≤ b.c  
   d. If a ≤ b and c < 0, then a.c ≥ b.c
4. Prove that  
    *for all a,b:* *if a > 0 and b > 0, then a.b > 0*  
   is true for all rational numbers a,b, using only the axioms for addition and multiplication, the order-axioms without iv), 0 < 1, and the fact that (\*) holds for all natural numbers.  
   (Hint: First prove that 0 < 1/n < 1 for all n ≥ 2.)
5. If a > b and c > 0 then a/c > b/c
6. i. If a > 0, then a2 > 0.  
   ii. If a < 0, then a2 > 0.
7. If a > 0, then (–a) . b = –(a.b)
8. If 0 < a < b, then 1/b < 1/a  
   (Bonus question: is the condition 0 < a needed?)
9. |a + b| ≤ |a| + |b| (“triangle inequality”, Dutch: driehoeksongelijkheid)
10. (a + b)2 = a2 + 2ab + b2
11. (a – b)2 = a2 – 2ab + b2
12. (a – b)(a + b) = a2 – b2
13. If 0 ≤ a < b, then a2 < b2  
    (Is the condition 0 ≤ a needed?)
14. (a.b)2 = a2.b2, (ab)3 = a3b3, etc.
15. a2 ≥ 0 for all a, |a|2 = a2 for all a
16. If 0 < a < b and 0 < c < d, then a.c < b.d
17. Define max(a,b) as the largest of a and b, min(a,b) as the smallest of a and b.   
    Prove that for all real numbers a,b: min(a,b) ≤ (a + b) / 2 ≤ max(a,b).
18. (\*) If 0 < a, 0 < b then 1/a + 1/b ≥ 4/(a+b)
19. If 0 ≤ a ≤ b, then 1/(a+1) ≥ 1/(b+1)  
    (Is 0 ≤ a needed?)
20. If a > b > 0, then a2/b2 > a/b
21. If a < b, then there is a x ∈ ℝ such that a < x < b
22. (\*) If a < b, then there is a q ∈ ℚ such that a < q < b
23. If 0 < a < b, then a2 < b2, a3 < b3, a4 < b4, etc  
    (Is 0 < a needed?)
24. (\*) (a + b)2 ≤ 2.(a2 + b2)  
    When is this inequality strict?

Exercise 2.6

a. Determine the truth table of the formula: (P ⇒ Q) ∨ (Q ⇒ P)

b. Investigate by using truth tables whether or not the following two formulas are logically

equivalent or not: (P ⇒ (Q ∨ R)) ; (P ⇒ Q) ∨ (P ⇒ R)

c. Are the following statements true for all sets A,B,C, or are there counterexamples?

- Either A ⊆ B or B ⊆ A

- If A ⊆ B ∪ C then either A ⊆ B or A ⊆ C

Exercise 2.7

For each of the following claims, determine if they are valid for all sets A, B, C or not. (And if not, provide a counterexample.)

a. A ⊆ B iff A ∪ B = B

b. A ⊆ B iff A ∩ B = A

c. A ∪ A = A ∩ A = A

d. A ∩ ∅ = ∅

e. A ∪ ∅ = A

f. If C ⊆ A and C ⊆ B, then C ⊆ A ∩ B

g. If C ⊆ A ∩ B, then C ⊆ A and C ⊆ B

h. If A ⊆ C and B ⊆ C, then A ∪ B ⊆ C

i. If A ∪ B ⊆ C, then A ⊆ C and B ⊆ C

j. A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

k. A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Exercise 2.8

What can you say about the following sets.

Can you formulate and prove a general law about them?

a. (A ∩ B) ∪ A = ?

b. (A ∪ B) ∩ A = ?

Exercise 2.9

Prove the following statements (or a sizable portion of them), for all sets A,B,C,D, or find a

counterexample:

a. A \ (A \ B) = A ∩ B

b. (A ∩ B) ∪ (A ∩ Bc) = A

c. A ∪ (B ∖ A) = A ∪ B

d. (A ∖ B) ∖ C = A ∖ (B ∪ C)

e. (A ∖ B) ∪ (B ∖ C) ∪ (C ∖ A) = (A ∪ B ∪ C) ∖ (A ∩ B ∩ C)

f. (A ∖ B)c = Ac ∪ B

g. (A ∖ (B ∪ C)) ∪ (A ∖ (B ∩ C)) = (A ∖ B) ∪ (A ∖ C)

h. (A \ (B \ C)) = (A \ B) ∪ (A ∩ C)

i. (A ∪ B) \ C = (A \ C) ∪ (B \ C)

j. (A \ (B ∩ C)) = (A \ B) ∪ (A \ C)

k. (A \ (B ∪ C)) = (A \ B) ∩ (A \ C)

l. (A ∩ B) ∪ (A ∩ C) ∪ (B ∩ C) = (A ∪ B) ∩ (B ∪ C) ∩ (C ∪ A)

m. (A ∪ B ∪ C) ∩ (A ∪ Bc ∪ C) ∩ (A ∪ B ∪ Cc) = (A ∪ (B ∩ C))

n. (A ∩ B) ∪ (A ∩ Bc ∩ C) = A ∩ (B ∪ C)

o. (A ∩ B) ∪ (B ∩ C) ∪ (C ∩ A) = (A ∪ B ∪ C) ∩ ((A ∩ B) ∪ (B ∩ C) ∪ (C ∩ A))

p. (A ∩ B) ∪ (B ∩ C) ∪ (C ∩ D) = (A ∪ B ∪ C) ∩ (B ∪ C ∪ D)

q. (A ∩ B) ∪ (C ∩ D) = (A ∪ C) ∩ (B ∪ D) If A = D en B = C

r. (A ∩ B) ∪ (C ∩ D) = (A ∪ C) ∩ (B ∪ D) if A = C en B = D

s. (A ∪ B) ∩ (B ∪ C) = (B ∪ A) ∩ (C ∪ A)

Exercise 2.10 (De Morgan Laws for sets)

Prove the following statements, and illustrate them with Venn diagrams:

a. (A ∩ B)c = Ac ∪ Bc

b. (A ∪ B)c = Ac ∩ Bc

Exercise 2.11

Prove the following statement, or find a counterexample (recall that the notation |…| is

used to denote the number of elements of a (finite) set :

If A and B are finite sets, then

| A ∪ B | + | A ∩ B | = |A| + |B|

Exercise 2.12

Prove the following statements, or find a counterexample:

a. ℘(A ∪ B) = ℘(A) ∪ ℘(B)

b. ℘(A ∩ B) = ℘(A) ∩ ℘(B)